Closing Thu: 10.3
Closing next Thu: 13.3(part 1)
Midterm 1 is Tuesday, Feb. 2 it covers
12.1-12.6, 10.1-10.3, 13.1-13.2

13.3(1) is about curvature.
(I will NOT ask about this on our midterm 1).

10.3 Polar Coordinates (continued)

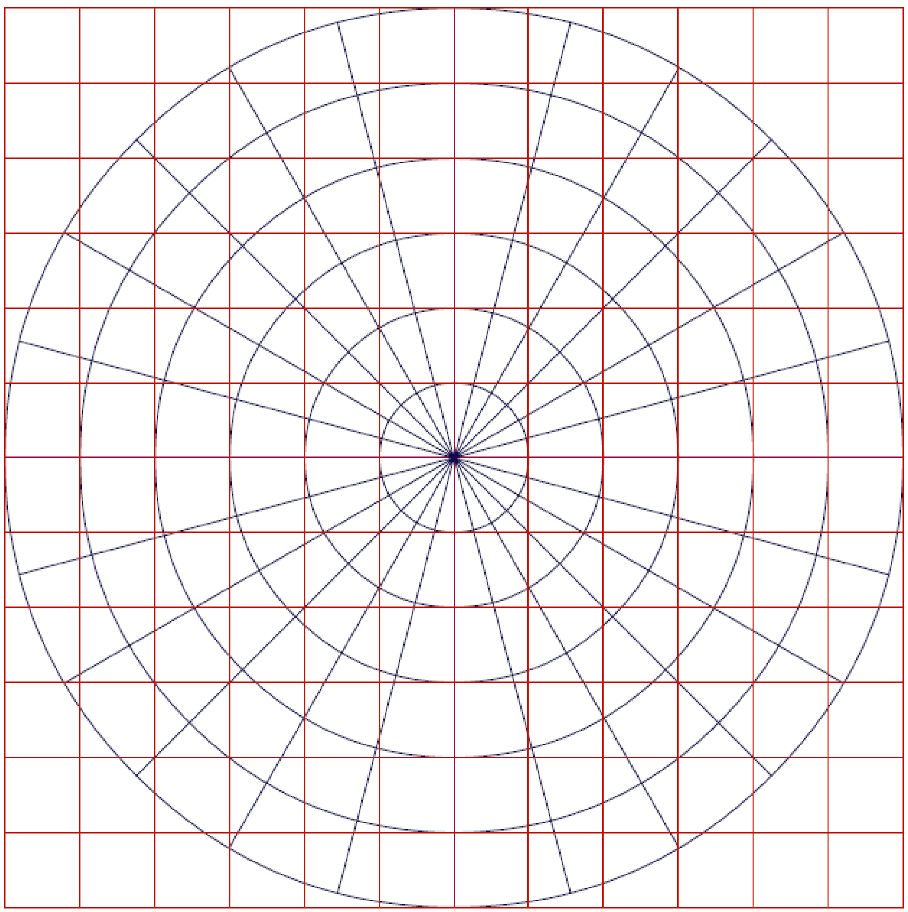
Entry Task:
By plotting the following points, graph

$r = 1 + \sin(\theta)$

θ	r		θ	r
0			π	
π/4			5π/4	
π/2			3π/2	
3π/4			7π/4	

Note: $1 + \sqrt{2}/2 \approx 1.71$

$1 - \sqrt{2}/2 \approx 0.29$



Finding dy/dx:

Recall, in polar we always know that
x = r cos(θ) and y = r sin(θ)

So, if $r = f(\theta)$,
then $x = rcos(\theta) = f(\theta)\cos(\theta)$
$y = rsin(\theta) = f(\theta)\sin(\theta)$
This is a parametric equation for x and y!

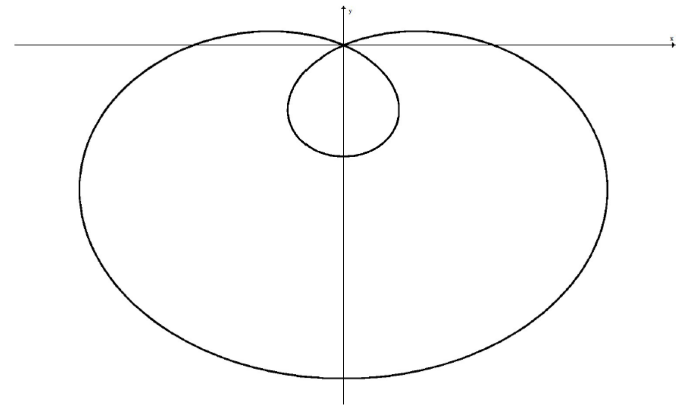
From what we learned about parametric equations:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin(\theta)+f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta)-f(\theta)\sin(\theta)}$$

which is often written as:

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin(\theta) + r\cos(\theta)}{\frac{dr}{d\theta}\cos(\theta) - r\sin(\theta)}$$

Example (from an old midterm):
Consider r = 3 – 6sin(θ) (shown below)

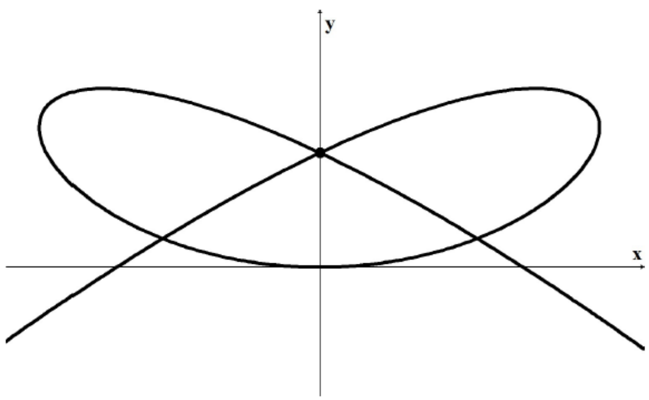


(a) Find the y-intercepts.
(b) Find the equation for the tangent line at θ = π.
(Put your answer in the form y = mx+b)

Parametric Example (more old exams):
Consider the curve given by

$$x = t^3 - 4t,\ y = 5t^2 - t^4$$

The curve intersects the positive y-axis at the same y-intercept twice. Find the two different tangent slopes at this point.



13.3 (part 1) Curvature

The **curvature** at a point, K, is a measure of how quickly a curve is changing direction at that point.

We want to define

$$K = \frac{change\ in\ direction}{change\ in\ arc\ length(distance)}$$

Roughly, how much does your direction change if you move "one inch" along the curve?

Let

$$\overrightarrow{\boldsymbol{T_1}} = unit\ direction\ vector\ at\ the\ point$$

$$\overrightarrow{\boldsymbol{T_2}} = unit\ direction\ vector\ one\ inch\ later$$

So

$$K \approx \left|\frac{\overrightarrow{\boldsymbol{T_2}} - \overrightarrow{\boldsymbol{T_1}}}{one\ inch}\right| = \left|\frac{\Delta\overrightarrow{\boldsymbol{T}}}{\Delta s}\right|$$

We define curvature to be the limit as the distance goes to zero, which gives

$$K = \left|\frac{d\overrightarrow{\boldsymbol{T}}}{ds}\right|$$

Example (the long way):

Consider x = t, y = cos(t), z = sin(t)

(1) Write the function for arc length
(2) Reparameterize in terms of arc length.
(3) Find the unit tangent with respect to *s*
(4) Find the curvature.

First Shortcut:

$$K = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \left|\frac{d\vec{\boldsymbol{T}}/dt}{ds/dt}\right| = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|}$$

Faster Shortcut:

$$K = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|} = \frac{|\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|^3}$$

Explanation of short cut:

First note: $\vec{\boldsymbol{T}}$ and $\vec{\boldsymbol{T}}'$ are always orthogonal.

Proof:

Since $\vec{\boldsymbol{T}} \cdot \vec{\boldsymbol{T}} = \left|\vec{\boldsymbol{T}}\right|^2 = 1$, we can differentiate both sides to get

$$\vec{\boldsymbol{T}}' \cdot \vec{\boldsymbol{T}} + \vec{\boldsymbol{T}} \cdot \vec{\boldsymbol{T}}' = 0.$$

So $2\vec{\boldsymbol{T}} \cdot \vec{\boldsymbol{T}}' = 0$ and $\vec{\boldsymbol{T}} \cdot \vec{\boldsymbol{T}}' = 0$.

Since $\vec{\boldsymbol{T}}(t) = \frac{\vec{\boldsymbol{r}}'(t)}{|\vec{\boldsymbol{r}}'(t)|}$, we can write

$$\vec{\boldsymbol{r}}'(t) = |\vec{\boldsymbol{r}}'(t)|\vec{\boldsymbol{T}}(t).$$

Differentiating and using the product rule:

$$\vec{\boldsymbol{r}}''(t) = |\vec{\boldsymbol{r}}'(t)|'\vec{\boldsymbol{T}}(t) + |\vec{\boldsymbol{r}}'(t)|\vec{\boldsymbol{T}}'(t).$$

Taking the cross-product of both sides with $\vec{\boldsymbol{T}}$:

$\vec{\boldsymbol{T}} \times \vec{\boldsymbol{r}}'' = |\vec{\boldsymbol{r}}'|' \left(\vec{\boldsymbol{T}} \times \vec{\boldsymbol{T}}\right) + |\vec{\boldsymbol{r}}'| \left(\vec{\boldsymbol{T}} \times \vec{\boldsymbol{T}}'\right)$, so

$$\frac{\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''}{|\vec{\boldsymbol{r}}'|} = |\vec{\boldsymbol{r}}'| \left(\vec{\boldsymbol{T}} \times \vec{\boldsymbol{T}}'\right),$$

(because $\vec{\boldsymbol{T}} \times \vec{\boldsymbol{T}} = < 0, 0, 0 >$, tell me why?)

taking the magnitude

$\frac{|\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|} = |\vec{\boldsymbol{r}}'| \left|\vec{\boldsymbol{T}} \times \vec{\boldsymbol{T}}'\right|$, and

$\frac{\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''}{|\vec{\boldsymbol{r}}'|^2} = \left|\vec{\boldsymbol{T}}\right|\left|\vec{\boldsymbol{T}}'\right| sin\left(\frac{\boldsymbol{\pi}}{\boldsymbol{2}}\right)$, tell me why?

Thus,

$$\left|\vec{\boldsymbol{T}}'\right| = \frac{\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''}{|\vec{\boldsymbol{r}}'|^2}$$

Therefore $K = \left|\frac{d\vec{\boldsymbol{T}}}{ds}\right| = \frac{|\vec{\boldsymbol{T}}'(t)|}{|\vec{\boldsymbol{r}}'(t)|} = \frac{|\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|^3}$

Note:

To find curvature for a function y = f(x) in 2D, we can form a 3D vector function

$$\vec{\boldsymbol{r}}(x) = \langle x, f(x), 0\rangle$$

so $\vec{\boldsymbol{r}}'(x) = \langle 1, f'(x), 0\rangle$ and

$$\vec{\boldsymbol{r}}''(x) = \langle 0, f''(x), 0\rangle$$

$$|\vec{\boldsymbol{r}}'(x)| = \sqrt{1 + (f'(x))^2}$$

$$\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}'' = \langle 0,0, f''(x)\rangle$$

Thus,

$$K(x) = \frac{|\vec{\boldsymbol{r}}' \times \vec{\boldsymbol{r}}''|}{|\vec{\boldsymbol{r}}'|^3} = \frac{|\boldsymbol{f}''(x)|}{\left(\mathbf{1} + \left(\boldsymbol{f}'(\boldsymbol{x})\right)^2\right)^{3/2}}$$